

To appear in J. Algebra

INTEGER-VALUED POLYNOMIALS ON ALGEBRAS

Sophie Frisch

frisch@TUGraz.at

ABSTRACT. Let D be a domain with quotient field K and A a D -algebra. A polynomial with coefficients in K that maps every element of A to an element of A is called integer-valued on A . For commutative A we also consider integer-valued polynomials in several variables. For an arbitrary domain D and I an arbitrary ideal of D we show I -adic continuity of integer-valued polynomials on A . For Noetherian one-dimensional D , we determine spectrum and Krull dimension of the ring $\text{Int}_D(A)$ of integer-valued polynomials on A . We do the same for the ring of polynomials with coefficients in $M_n(K)$, the K -algebra of $n \times n$ matrices, that map every matrix in $M_n(D)$ to a matrix in $M_n(D)$.

2000 MSC: Primary 13F20; Secondary 16S50, 13B25, 13J10, 11C08, 11C20.

1. Introduction

Let D be a domain with quotient field K and A a D -algebra, such as, for instance, a group ring $D(G)$ or the matrix algebra $M_n(D)$.

We are interested in the rings of polynomials

$$\text{Int}_D(A) = \{f \in K[x] \mid f(A) \subseteq A\},$$

and, if A is commutative,

$$\text{Int}_D^n(A) = \{f \in K[x_1, \dots, x_n] \mid f(A^n) \subseteq A\}.$$

Elements of the D -algebra A are plugged into polynomials with coefficients in K via the canonical homomorphism $\iota_A : A \rightarrow K \otimes_D A$, $\iota_A(a) = 1 \otimes a$.

In the special case $A = D$ these rings are known as rings of integer-valued polynomials, cf. [3]. They provide natural examples of non-Noetherian Prüfer rings [5, 11], and have been used for proving results on the n -generator property in Prüfer rings [2]. Also, integer-valued polynomials are useful for polynomial

interpolation of functions from D to D [8, 4], and satisfy other interesting algebraic conditions such as analogues of Hilbert's Nullstellensatz [3, 9].

These desirable properties of rings of integer-valued polynomials have motivated the generalization to polynomials with coefficients in K acting on a D -algebra A [10, 12]. So far, not much is known about rings of integer-valued polynomials on algebras. We know that they behave somewhat like the classical rings of integer-valued polynomials if the D -algebra A is commutative. For instance, Loper and Werner [12] have shown that $\text{Int}_{\mathbb{Z}}(\mathcal{O}_K)$ is Prüfer. If A is non-commutative, however, the situation is radically different. For instance, $\text{Int}_{\mathbb{Z}}(M_2(\mathbb{Z}))$ is not Prüfer [12], and is far from allowing interpolation [10].

We will describe the spectrum of $\text{Int}_D(A)$, for a one-dimensional Noetherian ring D and a finitely generated torsion-free D -algebra A , in the hope that this will facilitate further research. We will investigate more closely the special case of $A = M_n(D)$: we determine a polynomially dense subset of $\text{Int}_D(M_n(D))$ and describe the image of a given matrix under the ring $\text{Int}_D(M_n(D))$.

A different ring of integer-valued polynomials on the matrix algebra $M_n(D)$, consisting of polynomials with coefficients in $M_n(K)$ that map matrices in $M_n(D)$ to matrices in $M_n(D)$, has been introduced by Werner [13]. We will show that it is isomorphic to the algebra of $n \times n$ matrices over “our” ring $\text{Int}_D(M_n(D))$ of integer-valued polynomials on $M_n(D)$ with coefficients in K .

Before we give a precise definition of the kind of D -algebra A for which we will investigate $\text{Int}_D(A)$, a few examples. D is always a domain with quotient field K , and not a field.

1.1 Example. For fixed $n \in \mathbb{N}$, let $A = M_n(D)$ be the D -algebra of $n \times n$ matrices with entries in D and

$$\text{Int}_D(M_n(D)) = \{f \in K[x] \mid \forall C \in M_n(D) : f(C) \in M_n(D)\}.$$

1.2 Example. Let $H = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ be the \mathbb{Q} -algebra of rational quaternions, $L = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ the \mathbb{Z} -subalgebra of Lipschitz quaternions, and

$$\text{Int}_{\mathbb{Z}}(L) = \{f \in \mathbb{Q}[x] \mid \forall z \in L : f(z) \in L\}.$$

1.3 Example. Let G be a finite group, $K(G)$ and $D(G)$ the respective group rings, and

$$\text{Int}_D(D(G)) = \{f \in K[x] \mid \forall z \in D(G) : f(z) \in D(G)\}.$$

If G is commutative, we also consider

$$\text{Int}_D^n(D(G)) = \{f \in K[x_1, \dots, x_n] \mid \forall z \in D(G)^n : f(z) \in D(G)\}.$$

for $n \in \mathbb{N}$, where $D(G)^n = D(G) \times \dots \times D(G)$ denotes the Cartesian product of n copies of $D(G)$.

1.4 Example. Let $D \subseteq A$ be Dedekind rings with quotient fields $K \subseteq F$, and

$$\text{Int}_D^n(A) = \{f \in K[x_1, \dots, x_n] \mid f(A^n) \subseteq A\}.$$

1.5 Notation and conventions. Throughout this paper, D is a domain and not a field, K the quotient field of D , and A a *torsion-free* D -algebra *finitely generated* as a D -module. Since A is faithful, there is an isomorphic copy of D embedded in A by $d \mapsto d1_A$, and we may assume $D \subseteq A$.

Now let $B = K \otimes_D A$. The natural homomorphisms $\iota_K : K \rightarrow K \otimes_D A$, $\iota_K(k) = k \otimes 1$ and $\iota_A : A \rightarrow K \otimes_D A$, $\iota_A(a) = 1 \otimes a$, are injective, since A is a torsion-free D module. We identify K and A with their isomorphic copies in B , which allows us to evaluate polynomials with coefficients in K at arguments in A , and define

$$\text{Int}_D(A) = \{f \in K[x] \mid \forall a \in A : f(a) \in A\}$$

and for $n \in \mathbb{N}_0$

$$\text{Int}_D^n(A) = \{f \in K[x_1, \dots, x_n] \mid \forall a_1, \dots, a_n \in A : f(a_1, \dots, a_n) \in A\}.$$

To exclude pathological cases we require $K \cap A = D$.

1.6 Remark. Instead of $B = K \otimes_D A$, we could look at the canonically isomorphic $A_{K \setminus \{0\}}$, the ring of fractions of A with denominators in $K \setminus \{0\}$. The natural homomorphisms $\iota_K : K \rightarrow A_{K \setminus \{0\}}$ and $\iota_A : A \rightarrow A_{K \setminus \{0\}}$ then take the form $\iota_K(\frac{c}{d}) = \frac{c1_A}{d}$ and $\iota_A(a) = \frac{a}{1}$.

1.7 Convention regarding polynomials in several variables. For non-commutative A and $n > 1$, $\text{Int}_D^n(A)$ is a priori not closed under multiplication and therefore in general not a ring. With the exception of the following section on I -adic continuity, we will only consider polynomial functions in several variables if the D -algebra A is commutative.

From section 3 onward, statements about $\text{Int}_D^n(A)$ with unspecified n and A are meant as follows: if A is commutative, let $n \in \mathbb{N}_0$, if A is non-commutative, assume $n \leq 1$.

1.8 Remark. Note that $K \cap A = D$ implies

$$\text{Int}_D(A) \subseteq \text{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\}$$

and for commutative A , also

$$\text{Int}_D^n(A) \subseteq \text{Int}(D^n) = \{f \in K[x_1, \dots, x_n] \mid f(D^n) \subseteq D\},$$

and $\text{Int}_D^0(A) = K \cap A = D$.

2. Continuity

I -adic continuity of the function $f: D^n \rightarrow D$ arising from a classical integer-valued polynomial $f \in \text{Int}(D^n) = \{f \in K[x_1, \dots, x_n] \mid f(D^n) \subseteq D\}$ has been shown, for an arbitrary ideal I of an arbitrary domain D , in Prop. 1.4 of [4]. (The proof there is for one variable, but clearly generalizable to several variables.)

To establish I -adic continuity of integer-valued polynomials on algebras, we will briefly look at polynomials in several non-commuting variables. If our algebra A is non-commutative, this becomes necessary, even if we are only interested in integer-valued polynomials in one variable: if we consider $f(x + y) - f(y)$ as a polynomial in two variables, and we still want substitution of elements from A for x and y to be a homomorphism, we must turn to non-commuting variables.

2.1 Definition. Let D be a domain with quotient-field K . Let $K\langle x_1, \dots, x_n \rangle$ be the free associative K -algebra generated by x_1, \dots, x_n (in other words, the semigroup-ring $K(S)$, where S is the free semigroup generated by x_1, \dots, x_n). If A is a torsion-free D -algebra, we evaluate polynomials in $K\langle x_1, \dots, x_n \rangle$ at arguments in $B = K \otimes_D A$ and thus associate a polynomial function $f: B^n \rightarrow B$ to every $f \in K\langle x_1, \dots, x_n \rangle$. Such polynomials as map arguments in A^n to values in A we call integer-valued on A .

From the theory of PID-rings (polynomial identity rings), it is easy to garner non-trivial examples of integer-valued polynomials in several non-commuting variables. For instance, if p is prime and $n \geq 1$, then a polynomial in $\mathbb{Q}\langle x_1, \dots, x_{np} \rangle$, but not in $\mathbb{Z}\langle x_1, \dots, x_{np} \rangle$, that takes every np -tuple of $n \times n$ integer matrices to an integer matrix is

$$f(x_1, \dots, x_{np}) = \frac{1}{p} \sum_{\pi \in S_{np}} x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(np)}.$$

(This follows from [7] or [14].) In this paper, we will not consider polynomials in non-commuting variables except in the following theorem and its corollaries.

2.2 Theorem. *Let D be a domain with quotient-field K and A a torsion-free D -algebra. For every $f \in K\langle x_1, \dots, x_n \rangle$ integer-valued on A , the polynomial function $f: A^n \rightarrow A$ is uniformly I -adically continuous for every ideal I of D .*

Proof. Fix i and let d be the degree of f in x_i . We will show that for every $b \in I^d A$ and every $(a_1, \dots, a_n) \in A^n$,

$$(*) \quad f(a_1, \dots, a_i + b, \dots, a_n) - f(a_1, \dots, a_i, \dots, a_n) \in IA.$$

For $d = 0$, or if f is the zero-polynomial, this is obvious. The polynomial f is uniquely representable as $f = f_1 + f_2$, where x_i doesn't occur in f_1 , x_i occurs in every monomial in the support of f_2 , and f_2 has the same degree in x_i as f . Since $f_1 = f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$ and $f_2 = f - f_1$, both f_1 and f_2 are integer-valued. We can show $(*)$ separately for f_1 and f_2 . As $(*)$ holds for f_1 and arbitrary $b \in A$, we have reduced to the case $f = f_2$, i.e., when x_i occurs in every monomial in the support of f .

Also, it suffices to show $(*)$ for $b = t_d t_{d-1} \dots t_1 c$ with $t_k \in I$ and $c \in A$, because every element of $I^d A$ is a finite sum of elements of this form.

By considering

$$g(x_1, \dots, x_n, z) = f(x_1, \dots, x_i + z, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)$$

we can reduce our task to showing for every $g(x_1, \dots, x_n, z) \in K\langle x_1, \dots, x_n, z \rangle$ of degree $d \geq 1$ in z , which maps A^{n+1} to A , satisfies $g(x_1, \dots, x_n, 0) = 0$, and such that z occurs in every monomial in the support of g :

$$(**) \quad g(a_1, \dots, a_n, t_d \dots t_1 c) \in IA \quad \text{whenever} \quad t_1, \dots, t_d \in I \quad \text{and} \quad c \in A.$$

We show $(**)$ by induction on d . Let $d \geq 1$. Consider

$$h(x_1, \dots, x_n, y) = g(x_1, \dots, x_n, t_d y) - t_d^d g(x_1, \dots, x_n, y).$$

h satisfies $h(x_1, \dots, x_n, 0) = 0$, is of degree at most $d - 1$ in y , and y occurs in every monomial in the support of h . Now $h(a_1, \dots, a_n, t_{d-1} \dots t_1 c) \in IA$ for all $t_1, \dots, t_{d-1} \in I$ and $c \in A$, either by induction hypothesis or because h is the zero-polynomial. Therefore $g(a_1, \dots, a_n, t_d t_{d-1} \dots t_1 c) = h(a_1, \dots, a_n, t_{d-1} \dots t_1 c) + t_d^d g(a_1, \dots, a_n, t_{d-1} \dots t_1 c)$ is in IA , for all $t_1, \dots, t_d \in I$ and $c \in A$. \square

Returning to commuting variables, we conclude:

2.3 Corollary. *For any ideal I of D , and every $f \in \text{Int}_D^n(A)$ the function $f: A^n \rightarrow A$ is uniformly I -adically continuous.*

2.4 Corollary. *If M is a maximal ideal of D , \hat{A} the M -adic completion of A , and $f \in \text{Int}_D^n(A)$, then the function $f: A^n \rightarrow A$ defined by f extends uniquely to a M -adically continuous function $f: \hat{A}^n \rightarrow \hat{A}$.*

3. A few technicalities

This section contains lemmata needed for the investigation of the spectrum of $\text{Int}_D(A)$ and, for commutative A , of $\text{Int}_D^n(A)$. From now on, all statements about $\text{Int}_D^n(A)$ are subject to the convention: if A is non-commutative, assume $n \leq 1$.

3.1 Lemma. *Let D be a domain and P a finitely generated prime ideal of height 1. Then there exist $m \in \mathbb{N}$, $p \in P$ and $s \in D \setminus P$ such that $sP^m \subseteq pD$.*

Proof. In the localization D_P , P_P is the radical of (q) for every non-zero $q \in P_P$. Fix such a $q \in P_P$. Since P (and hence P_P) is finitely generated, there exists $m \in \mathbb{N}$ with $P_P^m \subseteq qD_P$ and in particular $P^m \subseteq qD_P$.

The ideal P^m is also finitely generated, by p_1, \dots, p_k , say. Let $a_i \in D_P$ with $p_i = qa_i$. By considering the fractions $q = p/t$ and $a_i = r_i/s_i$ (with $p \in P$, $r_i \in D$ and $t, s_i \in D \setminus P$), and setting $s = t \cdot s_1 \cdot \dots \cdot s_k$, we see that $sP^m \subseteq pD$ as desired. \square

3.2 Definition. If $S \subseteq B^n$ and $T \subseteq B$, let

$$\text{Int}_D^n(S, T) = \{f \in K[x_1, \dots, x_n] \mid f(S) \subseteq T\}.$$

3.3 Lemma. *Let D be a domain and P a finitely generated prime ideal of height 1. Then every prime ideal Q of $\text{Int}_D^n(A)$ with $Q \cap D = P$ contains $\text{Int}_D^n(A, PA)$.*

Proof. Let $f \in \text{Int}_D^n(A, PA)$. By dint of Lemma 3.1, there are $m \in \mathbb{N}$, $p \in P$ and $s \in D \setminus P$ such that $sP^m \subseteq pD$. Then $sf^m \in \text{Int}_D^n(A, pA) = p\text{Int}_D^n(A) \subseteq Q$. As Q is prime and $s \notin Q$, we conclude that $f \in Q$. \square

3.4 Lemma. *Let A be a D -algebra that is finitely generated as a D -module, M a maximal ideal of finite index in D and \hat{A} the M -adic completion of A . For all $a \in \hat{A}^n$, the ideal $(M\hat{A})_a = \{f \in \text{Int}_D^n(A) \mid f(a) \in M\hat{A}\}$ is of finite index in $\text{Int}_D^n(A)$.*

Proof. Since A is a finitely generated D -module, \hat{A} is a finitely generated \hat{D} -module. As \hat{M} is of finite index in \hat{D} , $\hat{A}/M\hat{A}$ is finite. Let $\text{Int}_D^n(A)(a)$ denote the image of a under $\text{Int}_D^n(A)$. Then $\text{Int}_D^n(A)(a)/(M\hat{A} \cap \text{Int}_D^n(A)(a))$, as a subring of $\hat{A}/M\hat{A}$, is finite.

Let b_1, \dots, b_m be a system of representatives of $\text{Int}_D^n(A)(a)$ modulo $M\hat{A} \cap \text{Int}_D^n(A)(a)$, and for $1 \leq i \leq m$ let $f_i \in \text{Int}_D^n(A)$ with $f_i(a) = b_i$. Then for every $f \in \text{Int}_D^n(A)$, exactly one of the differences $f - f_i$ is in $(M\hat{A})_a$, which means that $\{f_1, \dots, f_m\}$ is a complete system of residues of $\text{Int}_D^n(A)(a)$ modulo $(M\hat{A})_a$. We have shown $[\text{Int}_D^n(A) : (M\hat{A})_a] = [\text{Int}_D^n(A)(a) : M\hat{A} \cap \text{Int}_D^n(A)(a)] \leq [\hat{A} : M\hat{A}]$ \square

4. Primes lying over a height one maximal ideal of finite index.

In this section we determine the prime ideals of $\text{Int}_D(A)$ (and, for commutative A , of $\text{Int}_D^n(A)$) lying over a height one prime ideal of finite index of D . Prime ideals lying over a prime of infinite index in D will be characterized in the next section.

4.1 General hypotheses in this section.

- (i) D is a domain and A a torsion-free D -algebra, finitely generated as a D -module;
- (ii) M is a finitely generated maximal ideal of height 1 and finite index in D ;
- (iii) $MA_M \cap A = MA$. (Note that $MA_M \cap A = MA$ is satisfied in two important cases: if A is a free D -module, and if $D \subseteq A$ is an extension of Dedekind rings.)

We denote the M -adic completions of D and A by \hat{D} and \hat{A} . Since M is finitely generated, M -adic and \hat{M} -adic topologies coincide on \hat{D} and on \hat{A} .

4.2 Lemma. *The hypotheses of 4.1 imply*

- (iv) $M\hat{A} \cap A = MA$.
- (v) \hat{D} and \hat{A} are compact.

Proof. (iv) Whenever M is a finitely generated maximal ideal of height 1 in a domain D and A a finitely generated D -module, the equality $M\hat{A} \cap A = MA_M \cap A$ holds, by [1, Chapter III, §2.12, Prop. 16] combined with [1, §3.5, Cor. 1 of Prop. 9].

(v) The ring \hat{D} is compact because M is of finite index and finitely generated, which implies that all powers of M are of finite index. \hat{A} then is compact because it is finitely generated as a \hat{D} -module. \square

4.3 Notation. Recall our convention that we only allow $n > 1$ in $\text{Int}_D^n(A)$ if A is commutative; for non-commutative A , $n = 1$ is assumed.

The image of $a \in \hat{A}$ under $\text{Int}_D^n(A)$ and $\text{Int}_D(A)$ we denote as follows:

$$\text{Int}_D^n(A)(a) = \{f(a) \mid f \in \text{Int}_D^n(A)\} \quad \text{and} \quad \text{Int}_D(A)(a) = \{f(a) \mid f \in \text{Int}_D(A)\}.$$

If M is a maximal ideal of D and $a \in \hat{A}^n$ let

$$(M\hat{A})_a = \{f \in \text{Int}_D^n(A) \mid f(a) \in M\hat{A}\}.$$

If P is an ideal of a commutative ring between $\text{Int}_D^n(a)$ and \hat{A} , let

$$P_a = \{f \in \text{Int}_D^n(A) \mid f(a) \in P\}.$$

4.4 Lemma. *Under the hypotheses of 4.1, let Q be a prime ideal of $\text{Int}_D^n(A)$ lying over M . Then there exists $a \in \hat{A}$ such that*

$$(M\hat{A})_a \subseteq Q.$$

In particular, Q is of finite index.

Proof. Suppose Q does not contain any $(M\hat{A})_a$. Then for every $a \in \hat{A}^n$ there exists $f \in (M\hat{A})_a \setminus Q$. By Theorem 2.3, f is M -adically continuous, so there exists an M -adic neighborhood U of a , such that $f(U) \subseteq M\hat{A}$. By compactness of \hat{A}^n , there exist finitely many a_i such that the corresponding neighborhoods U_i cover \hat{A}^n . Let g be the product of the polynomials $f_i \in (M\hat{A})_{a_i} \setminus Q$ with $f_i(U_i) \subseteq M\hat{A}$. For all $a \in A$, $g(a) \in M\hat{A} \cap A = MA_M \cap A = MA$ (by Remark 4.2). Since Q is prime, $g \notin Q$, and yet $g \in \text{Int}_D^n(A, MA)$, a contradiction to Lemma 3.3. We have shown that Q contains some $(M\hat{A})_a$, which is of finite index by Lemma 3.4. \square

In the special case $A = D$, the preceding lemma already concludes the characterization of primes of $\text{Int}_D^n(D)$ lying above a maximal ideal M of finite index in D (a result of Chabert [6]), because then $(M\hat{A})_a$ is $(M\hat{D})_a = \{f \in \text{Int}_D^n(D) \mid f(a) \in \hat{M}\}$, a prime ideal of finite index, and hence maximal. Chabert, however, showed the other inclusion, $Q \subseteq (M\hat{A})_a$, after first showing independently that Q must be maximal [3, Prop. V.2.2].

4.5 Lemma. *Under the hypotheses of 4.1, let Q be a prime ideal of $\text{Int}_D^n(A)$ lying over M . For every $a \in \hat{A}$ such that $(M\hat{A})_a \subseteq Q$, there exists a maximal ideal P of $\text{Int}_D^n(A)(a)$ such that $Q = P_a$.*

Proof. Since $\hat{A}/M\hat{A}$ is a finite ring, $\text{Int}_D^n(A)(a)/(\text{Int}_D^n(A)(a) \cap M\hat{A})$ is a finite commutative ring. Let P_1, \dots, P_k be the maximal ideals of $\text{Int}_D^n(A)(a)$ containing $\text{Int}_D^n(A)(a) \cap M\hat{A}$.

Suppose Q is not contained in any $(P_i)_a$, for $1 \leq i \leq k$. Then, by prime avoidance, $Q(a) = \{f(a) \mid f \in Q\}$ is not contained in $\bigcup_{i=1}^k P_i$. Let $f \in Q$ such that $f(a)$ is not in any P_i . Then the residue class of $f(a)$ is a unit in $\text{Int}_D^n(A)(a)/(\text{Int}_D^n(A)(a) \cap M\hat{A})$.

Replacing f by a suitable power of f (using the fact that the group of units of $\text{Int}_D^n(A)(a)/(\text{Int}_D^n(A)(a) \cap M\hat{A})$ is finite) we see that there exists $f \in Q$ with $f(a) \equiv 1 \pmod{M\hat{A}}$. It follows that $1 - f \in (M\hat{A})_a \subseteq Q$ and therefore $1 \in Q$, a contradiction. \square

4.6 Theorem. *Let D be a domain, A a torsion-free D -algebra finitely generated as a D -module, M a finitely generated maximal ideal of D of finite index and height one, such that $MA_M \cap A = MA$.*

The prime ideals of $\text{Int}_D^n(A)$ lying over M are precisely the ideals of the form

$$P_a = \{f \in \text{Int}_D^n(A) \mid f(a) \in P\},$$

where $a \in \hat{A}$ (the M -adic completion of A), and P is a maximal ideal of $\text{Int}_D^n(A)(a)$ (the image of a under $\text{Int}_D^n(A)$) with $P \cap D = M$. In particular, all primes of $\text{Int}_D^n(A)$ lying over M are of finite index.

Proof. There exist primes of $\text{Int}_D^n(A)$ lying over M , because $(M\hat{A})_a$ with $a \in \hat{A}$ is a proper ideal of $\text{Int}_D^n(A)$ containing M .

If Q is a prime ideal of $\text{Int}_D^n(A)$ lying over M , then Lemma 4.4 shows that there exists an element $a \in \hat{A}$ such that $(M\hat{A})_a$ is contained in Q , and that Q is of finite index. It then follows from Lemma 4.5 that $Q = P_a$ for some maximal ideal P of $\text{Int}_D^n(A)(a)$ satisfying $M \subseteq \text{Int}_D^n(A)(a) \cap M\hat{A} \subseteq P$, and hence $P \cap D = M$.

Conversely, if P is a maximal ideal of $\text{Int}_D^n(A)(a)$, then $\text{Int}_D^n(A)/P_a$ is isomorphic to $\text{Int}_D^n(A)(a)/P$. Therefore P_a is a maximal ideal of $\text{Int}_D^n(A)(a)$. \square

It may happen that we do not know the exact image of $a \in \hat{A}$ under $\text{Int}_D^n(A)$, but do know a commutative ring R_a between $\text{Int}_D^n(A)(a)$ and \hat{A} . In this case we should remember that $R_a/(R_a \cap M\hat{A})$ is a subring of the finite ring $(\hat{A}/M\hat{A})$, and that therefore $\text{Int}_D^n(A)(a)/(\text{Int}_D^n(A)(a) \cap M\hat{A}) \subseteq (R_a/R_a \cap M\hat{A})$ is an extension

of finite commutative rings. Since extensions of finite commutative rings satisfy “lying over”, every prime ideal of $\text{Int}_D^n(A)(a)$ comes from a prime ideal of R_a , and we conclude:

4.7 Corollary. *Under the hypotheses of Theorem 4.6, suppose we have, for every $a \in \hat{A}$, a commutative ring R_a with $\text{Int}_D^n(A)(a) \subseteq R_a \subseteq \hat{A}$.*

Then the prime ideals of $\text{Int}_D^n(A)$ are precisely the ideals of the form

$$P_a = \{f \in \text{Int}_D^n(A) \mid f(a) \in P\},$$

where $a \in \hat{A}$ and P is a maximal ideal of R_a lying over M .

If A is a commutative D -algebra, we can take $R_a = \hat{A}$ in Corollary 4.7 for all $a \in \hat{A}$, and we get the following simpler characterization of the primes of $\text{Int}_D^n(A)$ lying over M :

4.8 Theorem. *Let D be a domain, A a commutative torsion-free D -algebra finitely generated as a D -module, M a finitely generated maximal ideal of D of finite index and height one, such that $MA_M \cap A = MA$.*

Then every prime ideal of $\text{Int}_D^n(A)$ lying over M is of the form

$$P_a = \{f \in \text{Int}_D^n(A) \mid f(a) \in P\},$$

for some $a \in \hat{A}$ (the M -adic completion of A) and P a maximal ideal of \hat{A} lying over M . In particular, every prime of $\text{Int}_D^n(A)$ lying over M is of finite index.

5. Primes lying over prime ideals of infinite index

5.1 Lemma. *Let A be a torsion-free D -algebra with $K \cap A = D$. Let P be a prime ideal of infinite index in D and $n \in \mathbb{N}$. Then $\text{Int}_D^n(A) \subseteq D_P[x_1, \dots, x_n]$.*

Proof. More generally, we show that for any prime ideal P , a polynomial $f \in \text{Int}_D^n(A)$ of degree less than $[D : P]$ in every individual variable is in $D_P[x_1, \dots, x_n]$. We use induction on n . The case $n = 0$ is trivial: $\text{Int}_D^0(D) = K \cap A = D \subseteq D_P$.

For $n > 0$ consider f as a polynomial in x_n with coefficients in $K[x_1, \dots, x_{n-1}]$. Let $s \leq [D : P]$ such that f is of degree strictly less than s in each x_j .

Choose $d_1, \dots, d_s \in D \subseteq A$ pairwise incongruent mod P . For every i , the value of f at d_i (substituted for x_n) is a polynomial in $\text{Int}_D^{n-1}(A)$ of degree less than

s in each variable. Therefore $f(x_1, \dots, x_{n-1}, d_i) \in D_P[x_1, \dots, x_{n-1}]$, by induction hypothesis.

Let $g \in K(x_1, \dots, x_{n-1})[x_n]$ be the Lagrange interpolation polynomial with $g(d_i) = f(x_1, \dots, x_{n-1}, d_i)$ ($1 \leq i \leq s$); then $g \in D_P[x_1, \dots, x_{n-1}][x_n]$. Since a polynomial (with coefficients in a domain) of degree less than s is determined by its values at s different arguments, we must have $f = g \in D_P[x_1, \dots, x_{n-1}][x_n]$. \square

5.2 Corollary. *Let A be a torsion-free D -algebra with $K \cap A = D$. If all maximal ideals of D are of infinite index, then $\text{Int}_D^n(A) = D[x_1, \dots, x_n]$.*

Alternatively, we could have deduced the previous lemma from the corresponding fact for the ring of integer-valued polynomials over D [3, Prop. I.3.4, XI.1.10], since after all $\text{Int}_D^n(A) \subseteq \text{Int}(D^n) = \{f \in K[x_1, \dots, x_n] \mid f(D^n) \subseteq D\}$.

5.3 Lemma. *Let A be a torsion-free D -algebra with $K \cap A = D$. Let P be a prime ideal of infinite index in D and $n \in \mathbb{N}$. Then the prime ideals of $\text{Int}_D^n(A)$ lying over P are precisely those of the form $Q \cap \text{Int}_D^n(A)$, where Q is a prime ideal of $D_P[x_1, \dots, x_n]$ containing $PD_P[x_1, \dots, x_n]$.*

Proof. As $D[x_1, \dots, x_n] \subseteq \text{Int}_D^n(A) \subseteq D_P[x_1, \dots, x_n] = D[x_1, \dots, x_n]_{(D \setminus P)}$, we have

$$D_P[x_1, \dots, x_n] = \text{Int}_D^n(A)_{(D \setminus P)}$$

and therefore a bijective correspondence (given by lying over) exists between prime ideals of $\text{Int}_D^n(A)$ whose intersection with D is contained in P and prime ideals of $D_P[x_1, \dots, x_n]$. \square

5.4 Theorem. *Let D be a Noetherian one-dimensional domain with finite residue fields and A a torsion-free D algebra, finitely generated as a D -module, such that for every maximal ideal M , $MA_M \cap D = M$. If A is commutative, let $n \in \mathbb{N}$, for non-commutative A restrict to $n = 1$. Then $\text{Int}_D^n(A)$ is $(n + 1)$ -dimensional.*

Proof. By Lemma 5.3, the prime ideals of $\text{Int}_D^n(A)$ lying over (0) all come from prime ideals of $K[x_1, \dots, x_n]$. The primes of $\text{Int}_D^n(A)$ lying over a maximal ideal M are all maximal and hence mutually incomparable. So $\dim(\text{Int}_D^n(A)) \leq n + 1$. If M is a maximal ideal of D and $d = (d_1, \dots, d_n) \in D^n$, let \mathcal{Q}_k be the ideal of $K[x_1, \dots, x_n]$ generated by $(x_1 - d_1), \dots, (x_k - d_k)$. Then a chain of primes of length $n + 1$ of $\text{Int}_D^n(A)$ is given by $Q_0 = (0)$, $Q_k = \mathcal{Q}_k \cap \text{Int}_D^n(A)$, for $k = 1, \dots, n$ and $Q_{n+1} = P_d = \{f \in \text{Int}_D^n(A) \mid f(d_1, \dots, d_n) \in P\}$, where P is a maximal ideal of the image of d under $\text{Int}_D^n(A)$. \square

5.5 Remark. If D is a Noetherian domain of characteristic 0, finitely generated as a \mathbb{Z} -algebra, such as, for instance, the ring of integers \mathcal{O}_K in a number field, then no maximal ideal of $\text{Int}_D^n(A)$ lies over (0) of D . This holds for $A = D$ by [3, Prop. XI.3.4.], and carries over to $\text{Int}_D^n(A)$, since $\text{Int}_D^n(A) \subseteq \text{Int}(D^n)$. Every prime ideal P of $\text{Int}_D^n(A)$ coming from a prime ideal of $K[x_1, \dots, x_n]$ is contained in a maximal ideal Q of $\text{Int}(D^n)$ lying over a maximal ideal M of D , and $Q \cap \text{Int}_D^n(A)$ then properly contains P .

5.6 Remark. Even if there are no maximal ideals lying over (0) , maximal chains of primes of $\text{Int}_D^n(A)$ are not necessarily of length $n + 1$. For instance, a maximal ideal of $\text{Int}_D^n(A)$ may well have height 1 if it is of the form $(M\hat{A})_a$ for $a = (a_1, \dots, a_n) \in \hat{A}^n$ with a_1, \dots, a_n algebraically independent over K .

6. Integer-valued polynomials on matrix algebras

Theorem 4.6 characterizes the spectrum of the ring $\text{Int}_D(A)$, provided we know the images of elements of M -adic completions of A under $\text{Int}_D(A)$. We will now determine these images in the case $A = M_n(D)$. Note that all the technical hypotheses in this section are certainly satisfied by the ring \mathcal{O}_K of integers in a number field.

6.1 Fact. [10, Lemma 2.2] *Let D be a domain and $f(x) = g(x)/d$ with $g \in D[x]$, $d \in D \setminus \{0\}$. Then $f \in \text{Int}_D(M_n(D))$ if and only if g is divisible modulo $dD[x]$ by all monic polynomials in $D[x]$ of degree n .*

6.2 Proposition. *Let D be a domain with zero Jacobson radical and $f(x) = g(x)/d$ with $g \in D[x]$, $d \in D \setminus \{0\}$. Then $f \in \text{Int}_D(M_n(D))$ if and only if g is divisible modulo $dD[x]$ by all monic irreducible polynomials in $D[x]$ of degree n .*

Proof. In view of Fact 6.1, it suffices to show for every $d \in D \setminus \{0\}$ and $h \in D[x]$ monic of degree n , that there exists $k \in D[x]$ monic of degree n , irreducible in $D[x]$ and congruent to $h \pmod{dD[x]}$. We may choose a maximal ideal P with $d \notin P$, and use Chinese remainder theorem on the coefficients of h to find $k \in D[x]$, monic of degree n , congruent to $h \pmod{dD[x]}$ and irreducible in $(D/P)[x]$. \square

We are now able to identify a polynomially dense subset of $M_n(D)$ consisting of companion matrices. They are often easier to work with than general matrices, because their characteristic polynomial is also their minimal polynomial.

6.3 Theorem. *Let \mathcal{C}_n be the set of companion matrices of monic polynomials of degree n in $D[x]$ and $\mathcal{I}_n \subseteq \mathcal{C}_n$ the subset of companion matrices of irreducible polynomials. If D is any domain,*

$$\text{Int}_D(M_n(D)) = \text{Int}_D(\mathcal{C}_n, M_n(D)).$$

If D is a domain with zero Jacobson radical, such as, for instance, a Dedekind domain with infinitely many maximal ideals, then

$$\text{Int}_D(M_n(D)) = \text{Int}_D(\mathcal{I}_n, M_n(D)).$$

Proof. Let $f \in \text{Int}_D(\mathcal{C}_n, M_n(D))$, $f(x) = g(x)/d$ with $g \in D[x]$, $d \in D$. Since g maps every $C \in \mathcal{C}_n$ to $M_n(dD)$, g is divisible mod $dD[x]$ by every monic polynomial in $D[x]$ of degree n . (This is so because f is still the minimal polynomial of its companion matrix when everything is viewed in D/dD .) By Fact 6.1, $f \in \text{Int}_D(M_n(D))$. This shows $\text{Int}_D(\mathcal{C}_n, M_n(D)) \subseteq \text{Int}_D(M_n(D))$. The reverse inclusion is trivial. The argument for \mathcal{I}_n is similar, using Prop. 6.2. \square

6.4 Theorem. *Let D be a domain and $C \in M_n(D)$. Let*

$$\text{Int}(A)(C) = \{f(C) \mid f \in \text{Int}_D(M_n(D))\} \quad \text{and} \quad D[C] = \{f(C) \mid f \in D[x]\}.$$

Then $\text{Int}(A)(C) = D[C]$.

Proof. Consider $f \in \text{Int}_D(M_n(D))$; $f(x) = g(x)/d$ with $g \in D[x]$ and $d \in D \setminus \{0\}$. We know that g is divisible modulo $dD[x]$ by every monic polynomial in $D[x]$ of degree n . Dividing g by χ_C , the characteristic polynomial of C , we get

$$g(x) = q(x)\chi_C(x) + dr(x)$$

with $q, r \in D[x]$ and we see that $f(C) = r(C)$. Thus $\text{Int}(A)(C) \subseteq D[C]$. The reverse inclusion is clear, since $D[x] \subseteq \text{Int}_D(M_n(D))$. \square

6.5 Definition. A local domain is called *analytically irreducible* if its completion is also a domain.

6.6 Lemma. *Let M be a maximal ideal of finite index in a domain D . Then the following are equivalent*

- (1) $\bigcap_{n=1}^{\infty} M^n = (0)$ and D_M is analytically irreducible
- (2) for every non-zero $d \in D$, cancellation of d is uniformly M -adically continuous.

Proof. ($1 \Rightarrow 2$) We have to show: for every non-zero $d \in D$, for every $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that for all $c \in D$: $dc \in M^k$ implies $c \in M^m$. Indirectly, suppose there exists $d \in D \setminus \{0\}$ and $m \in \mathbb{N}$, such that for every $k \in \mathbb{N}$ there is some $c_k \in D$ with $dc_k \in M^k$ and $c_k \notin M^m$. Since \hat{D} is compact and satisfies first countability axiom, (c_k) has a convergent subsequence. Its limit $c \in \hat{D}$ satisfies $c \notin M^m$, and hence $c \neq 0$, and also for all k , $dc \in M^k$, which implies $dc = 0$. We have shown the existence of zero-divisors in \hat{D} .

($2 \Rightarrow 1$) is easy. \square

6.7 Theorem. *Let D be a domain, M a maximal ideal of finite index of D such that $\bigcap_{n=1}^{\infty} M^n = (0)$ and D_M is analytically irreducible. Let \hat{D} be the M -adic completion of D , $C \in M_n(\hat{D})$, and $\text{Int}_D(M_n(D))(C)$ the image of C under $\text{Int}_D(M_n(D))$. Then*

$$\text{Int}_D(M_n(D))(C) \subseteq \hat{D}[C].$$

Proof. Let $f \in \text{Int}_D(M_n(D))$, $f(x) = g(x)/d$ with $g \in D[x]$, $d \in D$. For every $m \in \mathbb{N}$ let $k_m \in \mathbb{N}$ such that for all $c \in D$, $cd \in M^{k_m}$ implies $c \in M^m$. Let $E_1 = (e_{ij}^{(1)})$ and $E_2 = (e_{ij}^{(2)})$ be matrices in $M_n(D)$ with characteristic polynomials χ_1 and χ_2 . Then $g(x) = q_i(x)\chi_i(x) + dr_i(x)$ with $q_i, r_i \in D[x]$ for $i = 1, 2$. If $e_{ij}^{(1)} \equiv e_{ij}^{(2)} \pmod{M^{k_m}}$, then $dr_1 \equiv dr_2 \pmod{M^{k_m}D[x]}$, and therefore $r_1 \equiv r_2 \pmod{M^mD[x]}$. We can therefore M -adically approximate C by matrices C_i with $f(C_i) = s_i(C_i)$ with $s_i \in D[x]$, $\deg s_i < n$, such that the s_i converge towards a polynomial $s \in \hat{D}[x]$ with $\deg s < n$ and $f(C) = s(C)$. \square

6.8 Corollary. *Let D be a Dedekind domain, M a maximal ideal of finite index, \hat{D} the M -adic completion of D , and $C \in M_n(\hat{D})$. Then*

$$\text{Int}_D(M_n(D))(C) \subseteq \hat{D}[C].$$

7. Integer-valued polynomials with matrix coefficients

While we have been investigating the ring $\text{Int}_D(M_n(D))$ of polynomials in $K[x]$ mapping matrices in $M_n(D)$ to matrices in $M_n(D)$, Werner [13] has been studying the set, let's call it $\text{Int}_D[M_n(D)]$ with square brackets, of polynomials with coefficients in the non-commutative ring $M_n(K)$ mapping matrices in $M_n(D)$ to matrices in $M_n(D)$. Without substitution homomorphism, it is not a priori clear

that this set is closed under multiplication, but Werner [13] has shown that it is, and so $\text{Int}_D[M_n(D)]$ is actually a ring between $\text{Int}_D(M_n(D))$ and $\text{Int}_D(M_n(K))$.

Also in [13], Werner proves that every ideal of $\text{Int}_D[M_n(D)]$ can be generated by elements of $K[x]$. Using the idea of his proof, one can show more: $\text{Int}_D[M_n(D)]$ is isomorphic to the algebra of $n \times n$ matrices over $\text{Int}_D(M_n(D))$. Since every prime ideal of a matrix ring is just the set of matrices with entries in a prime ideal of the ring, we get a description of the spectrum of $\text{Int}_D[M_n(D)]$ as a byproduct of our description of the spectrum of $\text{Int}_D(M_n(D))$ in the previous section. We recall the definition of prime ideal for non-commutative rings:

7.1 Definition. We call a two-sided ideal $P \neq R$ of a (not necessarily commutative) ring with identity R a prime ideal, if, for all ideals A, B of R ,

$$AB \subseteq P \implies A \subseteq P \text{ or } B \subseteq P,$$

or equivalently, if, for all $a, b \in R$

$$aRb \subseteq P \implies a \in P \text{ or } b \in P.$$

For commutative R this is equivalent to the (in general stronger) condition: for all $a, b \in R$,

$$ab \in P \implies a \in P \text{ or } b \in P.$$

7.2 Theorem. Let D be a domain with quotient field K , and

$$\begin{aligned} \text{Int}_D(M_n(D)) &= \{f \in K[x] \mid \forall C \in M_n(D) : f(C) \in M_n(D)\} \\ \text{Int}_D[M_n(D)] &= \{f \in (M_n(K))[x] \mid \forall C \in M_n(D) : f(C) \in M_n(D)\}. \end{aligned}$$

We identify $\text{Int}_D[M_n(D)] \subseteq (M_n(K))[x]$ with its isomorphic image in $M_n(K[x])$ under

$$\varphi: (M_n(K))[x] \rightarrow M_n(K[x]), \quad \sum_k (a_{ij}^{(k)})_{1 \leq i, j \leq n} x^k \mapsto \left(\sum_k a_{ij}^{(k)} x^k \right)_{1 \leq i, j \leq n}$$

Then $\text{Int}_D[M_n(D)] = M_n(\text{Int}_D(M_n(D)))$.

Proof. Note that $K[x]$ is embedded in $M_n(K[x])$ as the subring of scalar matrices $g(x)I_n$, and in $M_n(K)[x]$ as the subring of polynomials $g(x)$ whose coefficients are scalar matrices rI_n , with $r \in K$. Clearly, $\text{Int}_D[M_n(D)] \cap K[x] = \text{Int}_D(M_n(D))$.

Let $C = (c_{ij}(x)) \in \text{Int}_D[M_n(D)] \subseteq M_n(K[x])$. Let e_{ij} be the matrix in $M_n(D)$ with 1 in position (i, j) and zeros elsewhere; then $e_{ij}Ce_{kl}$ has $c_{jk}(x)$ in position

(i, l) and zeros elsewhere. Also, $e_{ij}Ce_{kl} \in \text{Int}_D[M_n(D)]$, since $\text{Int}_D[M_n(D)]$ is a ring containing $M_n(D)$. So $\sum_{i=1}^n e_{ij}Ce_{ki} = c_{jk}(x)I_n \in \text{Int}_D[M_n(D)]$. Therefore $c_{jk}(x) \in K[x] \cap \text{Int}_D[M_n(D)] = \text{Int}_D(M_n(D))$ for all (j, k) , and hence $\text{Int}_D[M_n(D)] \subseteq M_n(\text{Int}_D(M_n(D)))$.

Conversely, if $f \in \text{Int}_D(M_n(D))$ then $f(x)I_n \in \text{Int}_D[M_n(D)]$. Therefore, $e_{ik}f(x)I_ne_{kl}$, the matrix containing $f(x)$ in position (i, l) and zeros elsewhere, is in $\text{Int}_D[M_n(D)]$, for arbitrary (i, l) . By summing matrices of this kind we see that $M_n(\text{Int}_D(M_n(D))) \subseteq \text{Int}_D[M_n(D)]$. \square

For any ring R with identity the ideals of R are in bijective correspondence with the ideals of $M_n(R)$ by $I \mapsto M_n(I)$, and restriction to prime ideals gives a bijection between the spectrum of R and the spectrum of $M_n(R)$. So we conclude:

- 7.3 Corollary.** *Let $\text{Int}_D(M_n(D))$ and $\text{Int}_D[M_n(D)]$ as in the preceding theorem. Under the identification of $\text{Int}_D[M_n(D)]$ with its isomorphic image in $M_n(K[x])$,*
- (1) *The two-sided ideals of $\text{Int}_D[M_n(D)]$ are precisely the sets of the form $M_n(I)$, where I is an ideal of $\text{Int}_D(M_n(D))$.*
 - (2) *The two-sided prime ideals of $\text{Int}_D[M_n(D)]$ are precisely the sets of the form $M_n(P)$, where P is a prime ideal of $\text{Int}_D(M_n(D))$.*

Acknowledgment. The author wishes to thank J.-L. Chabert for helpful hints, in particular Lemma 4.2 (iv), and constructive criticism.

References

- [1] N. BOURBAKI, *Commutative Algebra, Chapters 1–7*, Springer, 1989.
- [2] J. BOYNTON AND L. KLINGLER, *The n -generator property in rings of integer-valued polynomials determined by finite sets*, Arch. Math. (Basel) 86 (2006), 36–42.
- [3] P.-J. CAHEN AND J.-L. CHABERT, *Integer-valued polynomials*, vol. 48 of Mathematical Surveys and Monographs, Amer. Math. Soc., 1997.
- [4] P.-J. CAHEN, J.-L. CHABERT, AND S. FRISCH, *Interpolation domains*, J. Algebra 225 (2000), 794–803.
- [5] P.-J. CAHEN, J.-L. CHABERT, AND K. A. LOPER, *High dimension Prüfer domains of integer-valued polynomials*, J. Korean Math. Soc. 38 (2001), 915–935.

- [6] J.-L. CHABERT, *Les idéaux premiers de l'anneau des polynômes à valeurs entières*, J. Reine Angew. Math. 293/294 (1977), 275–283.
- [7] Q. CHANG, *A sort of polynomial identities of $M_n(F)$ with $\text{char } F \neq 0$* , Chinese Ann. Math. Ser. B 9 (1988), 161–166.
- [8] S. FRISCH, *Interpolation by integer-valued polynomials*, J. Algebra 211 (1999), 562–577.
- [9] S. FRISCH, *Nullstellensatz and Skolem properties for integer-valued polynomials*, J. Reine Angew. Math. 536 (2001), 31–42.
- [10] S. FRISCH, *Polynomial separation of points in algebras*, in Arithmetical Properties of Commutative Rings and Monoids (Chapel Hill Conf. 2003), S. Chapman, ed., Dekker, 2005, 249–254.
- [11] K. A. LOPER, *A classification of all D such that $\text{Int}(D)$ is a Prüfer domain*, Proc. Amer. Math. Soc. 126 (1998), 657–660.
- [12] K. A. LOPER AND N. J. WERNER, *Generalized rings of integer-valued polynomials*, J. Number Theory 132 (2012), 2481–2490.
- [13] N. J. WERNER, *Integer-valued polynomials over matrix rings*, Comm. Algebra 40 (2012), 4717 – 4726.
- [14] A. E. ZALESSKIĬ, *A symmetric analogue of the Amitsur-Levitzki theorem*, Vestsi Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk (1985), 108–110, 128.

Institut für Mathematik A
 Technische Universität Graz
 Steyrergasse 30
 A-8010 Graz, Austria
 frisch@TUGraz.at